

# G-THEORY OF $\mathbb{F}_1$ -ALGEBRAS I: THE EQUIVARIANT NISHIDA PROBLEM

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**ABSTRACT.** We develop a version of G-theory for  $\mathbb{F}_1$ -algebras and establish its first properties. We construct a Cartan assembly map to compare the Chu–Morava K-theory for finite pointed groups with our G-theory. We compute the G-theory groups for finite pointed groups in terms of stable homotopy of some classifying spaces. We also construct combinatorial Grayson operations on them. We discuss how our formalism is relevant to the Equivariant Nishida Problem - it asks whether  $\mathbb{S}^G$  admits operations that endow  $\bigoplus_n \pi_{2n}(\mathbb{S}^G)$  with a pre- $\lambda$ -ring structure, where  $G$  is a finite group and  $\mathbb{S}^G$  is the  $G$ -fixed point spectrum of the equivariant sphere spectrum.

## Introduction

Tits envisaged a geometry over "a field with one element  $\mathbb{F}_1$ " (also known at the absolute point) in [57], which has seen a resurgence in recent years. Building upon some earlier work of Soulé [53], Connes–Consani–Marcolli initiated a programme of developing  $\mathbb{F}_1$ -geometry with applications to algebraic geometry and arithmetic [13, 12, 11]. Several other noteworthy points-of-view on  $\mathbb{F}_1$  and important results have appeared in the literature, for instance, those of Borger [7], Toën–Vaquié [58], Durov [20], Deitmar [15], and so on. Instead of reproducing them here, we refer the readers to a recent survey article [38]. We also mention the articles [40, 41], which give a sense of the range and scope of such ideas. Our point of contact with  $\mathbb{F}_1$ -geometry is an early observation of Manin [39] - one interpretation of the Barratt–Priddy–Quillen Theorem is that the K-theory groups of  $\mathbb{F}_1$  are isomorphic to the stable homotopy groups of spheres. This was, according to the author, the first indication that  $\mathbb{F}_1$ -geometry is related to stable homotopy theory. Substantial work has been done on the K-theory of monoids [10], schemes [9] and Hochschild cohomology thereof [6]. In the world of stable homotopy theory some interesting connections with  $\mathbb{F}_1$ -geometry can be found in [47, 1]. There is a close cousin of K-theory, which is called G-theory (see [45] where it is called K'-theory), and the two theories are related by a natural Cartan homomorphism  $K_n(-) \rightarrow G_n(-)$ . In this article we develop a version of G-theory for  $\mathbb{F}_1$ -algebras using Waldhausen's K-theory of spaces [61]. The article is organised as follows:

In Section 1 we recall some basic facts about Waldhausen K-theory and modules over  $\mathbb{F}_1$ -algebras. We follow the approach of [11] and take an  $\mathbb{F}_1$ -algebra to simply mean a pointed monoid. In section 2, using Waldhausen's machinery, we define the G-theory spectrum  $\mathbf{G}(-)$  of an  $\mathbb{F}_1$ -algebra and set up some basic formal properties, like functoriality, transfer maps, etc.. Let  $G$  be a finite group and  $G_+$  denote the associated  $\mathbb{F}_1$ -algebra with a disjoint zero element. Let  $\mathbb{S}$  denote the sphere spectrum and let  $\mathbb{S}^G$  denote the model for the  $G$ -fixed point spectrum of the equivariant sphere spectrum obtained by Segal's  $\Gamma$ -space machine

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[50] applied to finite pointed  $G$ -sets. We show that there is a weak equivalence of spectra between  $\mathbb{S}^G$  and  $\mathbf{G}(G_+)$ . The spectrum  $\mathbf{G}(G_+)$  carries a natural multiplicative structure and we show that  $\pi_0(\mathbf{G}(G_+)) \cong A(G)$ , i.e., the Burnside ring of  $G$ . We also establish a connection between the  $G$ -theory of  $G_+$  and Waldhausen's  $\mathcal{A}$ -theory for  $BG$ . In Section 3 we develop some further properties of  $G$ -theory of  $G_+$ , like Mackey and Green structures. Such structures are quite useful for computational purposes - thanks to Axiomatic Induction Theory of Dress [17], they are often "hyperclementary computable". We construct a *Cartan assembly map*

$$BG_+ \wedge \mathbb{S} \rightarrow \mathbf{G}(G_+),$$

which compares the Chu–Morava K-theory theory with our  $G$ -theory (for a pointed finite group). The construction of this assembly map relies on the machinery of  $G$ -homology theories, as developed by Davis–Lück [14]. Motivated by the construction of the Whitehead spectrum of a group by Loday [37], we define the homotopy cofibre of the Cartan assembly map to be the ( $G$ -theoretic) *Loday–Whitehead spectrum* of  $G$  over  $\mathbb{F}_1$ . There is a map of spectra  $\mathbf{G}(G_+) = \mathbf{G}(\mathbb{F}_1[G]) \rightarrow \mathbf{K}(\mathbb{F}_q[G])$ , whenever the order of  $G$  is invertible in  $\mathbb{F}_q$ , which induces a map between the Loday–Whitehead groups  $\mathrm{Wh}_n(G, \mathbb{F}_1) \rightarrow \mathrm{Wh}_n(G, \mathbb{F}_q)$ . Since the groups  $\mathrm{Wh}_n(G, \mathbb{F}_q)$  are fairly computable, these homomorphisms give some information about the Loday–Whitehead groups of  $G$  over  $\mathbb{F}_1$ . Finally in Section 4, using Grayson's technology [26] that was enhanced by Gunnarsson–Schwänzl [27], we construct *combinatorial Grayson operations* on the  $G$ -theory of  $G_+$ . We also show that on  $G_0(G_+) \cong A(G)$  these operations recover Siebeneicher's pre- $\lambda$ -ring structure on the Burnside ring of a finite group [52]. Since  $\mathbf{G}(\mathbb{F}_1)$  is homotopy equivalent to  $\mathbb{S}$ , these operations can be viewed as operations on stable homotopy.

It follows from Serre's work on the unstable homotopy groups of spheres that  $\pi_n(\mathbb{S})$  is a finite abelian group for all  $n \geq 1$ . A celebrated result of Nishida says that all elements of  $\bigoplus_{n \geq 1} \pi_n(\mathbb{S})$  are nilpotent [43], which was conjectured earlier by Barratt. Iriye generalized the result by showing that every torsion element in  $\bigoplus_n \pi_n(\mathbb{S}^G)$  is nilpotent [33]. Note that  $\pi_0(\mathbb{S}^G)$  is the Burnside ring of  $G$ , which is torsion free. One consequence of this result is that the torsion elements in  $\bigoplus_n G_n(G_+)$  are all nilpotent. We formulate the following Question:

**Question 0.1** (Equivariant Nishida Problem). *Are there operations on  $\mathbb{S}^G$  that endow  $\bigoplus_n \pi_{2n}(\mathbb{S}^G)$  with a pre- $\lambda$ -ring structure?*

Nilpotence in stable homotopy theory is a very important topic (see, e.g., [46]). Ravenel formulated some striking conjectures, which were corroborated by Nishida's result. Most of Ravenel's conjectures were eventually solved by some ingenious methods [16, 31]. The exterior power functor gives rise to a  $\lambda$ -structure on complex topological K-theory. The  $\lambda$ -structure canonically gives rise to Adams operations, which turned out to be extremely useful for various purposes. Adams, Atiyah, Quillen, amongst others, used these operations very profitably to solve some problems in topology and representation theory. We believe that operations on equivariant stable homotopy are similarly interesting in their own right. Segal demonstrated the importance of such operations (known as total Segal operations) in [51]. Now we demonstrate how an affirmative answer to the above question implies the Iriye–Nishida result. J. Morava informed the author that G. Segal had already suggested this method to detect nilpotence and no claim to originality is made here. The author also understands that Morava–Santhanam have been independently working on operations in equivariant stable homotopy.

Let  $\oplus \pi_{2n}(\mathbb{S}^G)$  denote the even graded part, which is an honest commutative ring with identity. In order to prove the nilpotence of every torsion element in  $\oplus_n \pi_n(\mathbb{S}^G)$ , it suffices to show that every torsion element in the even graded part  $\oplus \pi_{2n}(\mathbb{S}^G)$  is nilpotent. Indeed, if  $x \in \pi_{2m+1}(\mathbb{S}^G)$  is a homogeneous torsion element of degree  $(2m+1)$ , then  $x^2$  is a homogeneous torsion element of degree  $(4m+2)$  and hence nilpotent. Since a finite sum of torsion nilpotent elements is again torsion nilpotent, the assertion follows. Any torsion element in a pre- $\lambda$ -ring is known to be nilpotent (see, e.g., Lemma on page 295 of [18]). Therefore, an affirmative answer to the Equivariant Nishida Problem recovers the Iriye–Nishida result on the nilpotence of torsion elements in  $\oplus_n \pi_n(\mathbb{S}^G)$ .

**Remark 0.2.** *Our result above concerning the pre- $\lambda$ -ring structure on the Burnside ring via combinatorial Grayson operations might look promising at first sight, but actually it is deficient. While the author believes that these operations on  $G$ -theory are inherently interesting, a closer inspection will reveal that they will not produce the desired pre- $\lambda$ -ring structure on  $\oplus_n G_{2n}(G_+)$ . Indeed, the combinatorial Grayson operations are maps  $\omega^k : G_n(G_+) \rightarrow G_n(G_+)$ . One can now readily verify that  $\omega^k(x + y) = \sum_{i=0}^k \omega^i(x) \omega^{k-i}(y)$  (one of the requirements in a pre- $\lambda$ -ring structure), will not be satisfied purely from degree considerations by setting, e.g.,  $x = y$  a homogeneous element in  $\oplus_{n \geq 1} G_{2n}(G_+)$  (unless the product structure is degenerate). However, a variant of the total Segal operation [51, 60], which can also be constructed on the  $G$ -theory of  $G_+$  thanks to [27], has a fair chance to work. It is also plausible that the answer to our question lies in [4].*

**Notations and conventions:** Unless otherwise stated, a (pointed) monoid or a ring is assumed to be unital but not necessarily commutative. By a module we always mean a right unital module. If  $M$  is an unpointed monoid, then its associated  $\mathbb{F}_1$ -algebra with a disjoint zero or absorbing element is typically denoted by  $\mathbb{F}_1[M]$  [11]. For notational simplicity, we denote it by  $M_+$ . For a pointed object, the basepoint is referred to as  $0$  or  $\star$ . Strictly speaking, the K-theory functor should be applied only to a small Waldhausen category. We ignore such set theoretic issues, since every Waldhausen category considered here has an evident small skeleton. Although not essential for the purposes of this paper, with some foresight, occasionally we work with a specific model for spectra, called *symmetric spectra* [32]. For such spectra there are two different homotopy groups - the *naïve* and the *true* ones. It is known that for *semistable* symmetric spectra (see Definition 5.6.1. of *ibid.*) the two possible homotopy groups agree (see [49] for an elaborate discussion). Since all symmetric spectra in sight will be semistable, we do not belabour this point here.

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## 1. SOME PRELIMINARIES

We recall some basic facts about Waldhausen K-theory and modules over  $\mathbb{F}_1$ -algebras.

**1.1. Waldhausen K-theory.** Waldhausen defined a K-theory functor, which is well suited for nonadditive situations. The functor takes a category with cofibrations and weak equivalences or a *Waldhausen category* as input and gives back a spectrum, whose homotopy groups are defined to be the K-theory groups of the Waldhausen category [61].

A *category with cofibrations* is a category  $\mathcal{C}$  with a chosen zero object  $\star$  and a subcategory  $\mathcal{cC}$ , whose morphisms are called *cofibrations* (denoted by  $C \rightarrowtail D$ ), satisfying the following:

- (1) the isomorphisms in  $\mathcal{C}$  belong to  $\mathcal{cC}$ ,
- (2) for any object  $C \in \mathcal{C}$ , the unique arrow  $\star \rightarrow C$  belongs to  $\mathcal{cC}$ , and
- (3) if  $C \rightarrowtail D$  is a cofibration and  $C \rightarrow B$  any arrow then the pushout  $D \amalg_C B$  exists in  $\mathcal{C}$  and the canonical map  $B \rightarrow D \amalg_C B$  is again a cofibration.

A *Waldhausen category*  $\mathcal{C}$  is a category with a chosen zero object  $\star$ , equipped with two distinguished subcategories  $\mathcal{cC}$  and  $w\mathcal{C}$ , whose arrows are called *cofibrations* and *weak equivalences* respectively. The data should satisfy some further axioms for which we refer the readers to Section 1.2 of [61]. For our purposes it suffices to say that any category with cofibrations  $\mathcal{C}$  as described above gives rise to a Waldhausen category by declaring the isomorphisms in  $\mathcal{C}$  to be the weak equivalences. The pushout of the cofibration  $C \rightarrowtail D$  along the map  $C \rightarrow \star$  will be referred to as the quotient  $D/C$ . Diagrams of the form  $C \rightarrowtail D \rightarrow D/C$  are called *cofibration sequences* and they play the role of ‘exact sequences’ in this nonadditive setting.

### Example 1.1.

- (i) (additive) The category of finitely generated modules over a Noetherian unital ring, where the cofibrations are monomorphisms and weak equivalences are isomorphisms.
- (ii) (additive) The category of perfect complexes over a unital ring, where the cofibrations are monomorphisms which are split in each degree, and the weak equivalences are quasi-isomorphisms of complexes.
- (iii) (nonadditive) The category of finite pointed simplicial sets, where the cofibrations are injective simplicial maps and the weak equivalences are simplicial weak equivalences.

A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  between Waldhausen categories is called *Waldhausen exact* or simply *exact* if it preserves all the structures, i.e., zero objects, cofibrations and weak equivalences, and whenever  $C \rightarrowtail D$  is a cofibration and  $C \rightarrow B$  any arrow in  $\mathcal{C}$ , the canonical map below is an isomorphism:

$$F(D) \amalg_{F(C)} F(D) \rightarrow F(D \amalg_C B)$$

Let  $\mathbf{Wald}$  denote the category of (small) Waldhausen categories with Waldhausen exact functors as morphisms. There is an  $\mathcal{S}_\bullet$ -construction, which produces a simplicial object in the category of Waldhausen categories with exact functors, i.e.,  $\mathcal{S}_\bullet : \mathbf{Wald} \rightarrow \mathbf{Wald}^{\Delta^{op}}$ . The  $n$ -simplices  $\mathcal{S}_n \mathcal{C}$  are  $n$ -step cofibration diagrams

$$C_\bullet := \star = C_{0,0} \rightarrowtail C_{0,1} \rightarrowtail \cdots \rightarrowtail C_{0,n}$$

with explicit choices of the quotients  $C_{i,j} = C_{0,j}/C_{0,i}$ . The morphisms  $C_\bullet \rightarrow D_\bullet$  are morphisms  $C_{i,j} \rightarrow D_{i,j}$  for all  $i \leq j$  which combine to form a morphism of diagrams in  $\mathcal{C}$ . In

particular,  $\mathcal{S}_0\mathcal{C}$  is the trivial category with only the zero object  $\star$  and the zero morphism, and  $\mathcal{S}_1\mathcal{C}$  is isomorphic to  $\mathcal{C}$ .

The weak equivalences in the category  $\mathcal{S}_n\mathcal{C}$  are those, where each  $C_{i,j} \rightarrow D_{i,j}$  is a weak equivalence in  $\mathcal{C}$ . The category  $w\mathcal{S}_\bullet\mathcal{C}$  is a simplicial category, whose category of  $n$ -simplices is  $w\mathcal{S}_n\mathcal{C}$ . By taking the nerve  $\mathcal{N}$  of the simplicial category  $w\mathcal{S}_\bullet\mathcal{C}$  levelwise one obtains a bisimplicial set  $\mathcal{N}w\mathcal{S}_\bullet\mathcal{C}$ . Now the Waldhausen  $K_i$ -group of  $\mathcal{C}$  is defined to be the homotopy group  $\pi_i$  of the based loop space of the geometric realization of this bisimplicial set, i.e.,  $\pi_i(\Omega|\mathcal{N}w\mathcal{S}_\bullet\mathcal{C}|)$ . For brevity, we write  $|\mathcal{N}w\mathcal{S}_\bullet\mathcal{C}|$  simply as  $|w\mathcal{S}_\bullet\mathcal{C}|$ . Waldhausen produced infinite deloopings  $\Omega|w\mathcal{S}_\bullet\mathcal{C}| \xrightarrow{\sim} \Omega^2|w\mathcal{S}_\bullet\mathcal{S}_\bullet\mathcal{C}| \xrightarrow{\sim} \Omega^3|w\mathcal{S}_\bullet\mathcal{S}_\bullet\mathcal{S}_\bullet\mathcal{C}| \xrightarrow{\sim} \cdots$  to exhibit a (connective) spectrum structure. In fact, Waldhausen K-theory defines a functor from **Wald** to the category of spectra, such that any natural isomorphism between exact functors induces a homotopy between maps of spectra (stated explicitly, e.g., in 1.5.3. and 1.5.4. of [56]). Later on in subsection 2.2 we shall see that the K-theory spectrum actually admits a symmetric spectrum structure in the sense of [32].

### Example 1.2.

- (1) Thanks to the work of Gillet–Waldhausen [24], the Waldhausen K-theory of Example (i) produces the G-theory of that ring.
- (2) It is known that if one feeds into this machine the Waldhausen category of Example (ii) above, then one recovers Quillen’s algebraic K-theory of that ring (see, e.g., Lemma 1.1. of [62]).
- (3) The Waldhausen K-theory of the Example (iii) above is called the  $\mathcal{A}$ -theory of a point.

**1.2. Modules over  $\mathbb{F}_1$ -algebras:** There seems to be a general consensus that a right (resp. left) module over an  $\mathbb{F}_1$ -algebra (or a pointed monoid)  $M$  should simply be a pointed set  $S$  with a pointed monoid homomorphism  $M^{\text{op}} \rightarrow \text{End}(S)$  (resp.  $M \rightarrow \text{End}(S)$ ). Here pointed homomorphism means that  $0 \in M$  must be sent to the endomorphism  $S \mapsto 0_S$  for all  $S \in S$ . A pointed map  $f : S_1 \rightarrow S_2$  between  $M$ -modules is called an  *$M$ -module homomorphism* if and only if  $f(sm) = f(s)m$  for all  $m \in M$  and  $s \in S_1$ . If there is a finite subset  $S' \subset S$ , such that  $\cup_{s \in S'} sM = S$ , then it is called *finitely generated*. Intuitively, the elements of  $S'$  form an  $M$ -generating set.

**Remark 1.3.** It is clear that if  $M$  is finite then any finitely generated  $M$ -module is also finite. In the absence of a additive structure we are not allowed to take finite linear combinations of elements of  $S$  with coefficients in  $M$ .

The category of unitary modules over a unital ring is additive, where finite coproduct is isomorphic to the finite product and typically denoted by the direct sum  $\oplus$ . In the category of  $M$ -modules, where  $M$  is an  $\mathbb{F}_1$ -algebra, the coproduct does not agree with the product. For example, the product of two  $\mathbb{F}_1$  modules (or pointed sets)  $S_1, S_2$  is given by the smash product  $S_1 \wedge S_2 := S_1 \times S_2 / \{S_1 \times 0_{S_2} \amalg 0_{S_1} \times S_2\}$ , whereas the coproduct is given by the pointed union, i.e.,  $S_1 \vee S_2 := S_1 \times 0_{S_2} \cup 0_{S_1} \times S_2$  with the canonical induced  $M$ -actions.

For any indexing set  $I$  we call an  $M$ -module  $F^I$ , equipped with a set map  $i : I \rightarrow F^I$ , *free on  $I$*  if and only if given any set map  $\iota : I \rightarrow S$ , where  $S$  is any  $M$ -module, there is a unique  $M$ -module map  $h : F^I \rightarrow S$ , such that  $hi = \iota$ . With this definition the module  $\bigvee_{j=1}^n M_j$ , where  $M_j = M$  for all  $j$ , is a finitely generated and free  $M$ -module. Here the indexing set is  $I = \{1, \dots, n\}$  and the map  $i : I \rightarrow \bigvee_{j=1}^n M_j$  is defined as  $i \mapsto 1_{M_i}$ .

In order to study K-theory one needs to work with the category of finitely generated and projective modules. The notion of projectivity is rather delicate in a nonadditive situation. Presumably a lifting property can lead to a good definition, i.e., an  $M$ -module  $S$  is *projective* if and only if given any surjective (at the level of pointed sets) map of  $M$ -modules  $f : S_1 \rightarrow S_2$  and any  $M$ -module map  $g : S \rightarrow S_2$ , there is a (not necessarily unique) lifting  $M$ -module map  $h : S \rightarrow S_1$  such that  $fh = g$ , e.g., in [15] Deitmar defines K-theory of  $\mathbb{F}_1$ -algebras along these lines. A more general theory has been developed by Chu–Lorscheid–Santhanam [9].

There is yet another theory over Noetherian rings, called G-theory, which uses the entire category of finitely generated modules. There is a natural Cartan homomorphism  $K_*(-) \rightarrow G_*(-)$  relating the two theories, which is an isomorphism for a regular Noetherian ring. In the next section we develop a version of G-theory for  $\mathbb{F}_1$ -algebras.

## 2. G-THEORY OF $\mathbb{F}_1$ -ALGEBRAS

For any  $\mathbb{F}_1$ -algebra  $M$ , let  $\mathbf{FG}(M)$  denote the category of finitely generated  $M$ -modules with  $M$ -module homomorphisms. Let us define the cofibrations in  $\mathbf{FG}(M)$  as those split monomorphisms of  $M$ -modules  $f : S_1 \rightarrow S_2$  (i.e., there is an  $M$ -module map  $\sigma : S_2 \rightarrow S_1$  with  $\sigma f = \text{id}_{S_1}$ ), such that  $S_2/S_1$  lies in  $\mathbf{FG}(M)$ . Here  $S_2/S_1$  denotes the pushout of the diagram  $S_2 \leftarrow S_1 \rightarrow *$  in  $\mathbf{FG}(M)$ , where  $*$  is the trivial  $M$ -module.

**Lemma 2.1.** *Endowed with the cofibrations just described,  $\mathbf{FG}(M)$  becomes a category with cofibrations.*

*Proof.* The only non-trivial condition that needs to be checked is (3). The pushout  $S_1 \coprod_S S_2$  of the diagram  $S_1 \xleftarrow{f_1} S \xrightarrow{f_2} S_2$  is constructed explicitly as  $S_1 \coprod S_2 / \{f_1(s) \sim f_2(s) \mid \forall s \in S\}$ . Let  $\sigma : S_1 \rightarrow S$  be the splitting of  $f_1$ . The splitting of the canonical map  $S_2 \rightarrow S_1 \coprod_S S_2$  is given by the map, which sends

$$\begin{aligned} s &\mapsto f_2 \circ \sigma(t) && \text{if } s \sim t \in S_1, \\ s &\mapsto s && \text{otherwise.} \end{aligned}$$

It is readily verified that  $(S_1 \coprod_S S_2)/S_2$  lies in  $\mathbf{FG}(M)$ . □

Now we promote  $\mathbf{FG}(M)$  to a Waldhausen category by setting the weak equivalences to be the isomorphisms in  $\mathbf{FG}(M)$ .

**Definition 2.2.** *The G-theory spectrum of an  $\mathbb{F}_1$ -algebra  $M$ , denoted by  $\mathbf{G}(M)$ , is defined to be the Waldhausen K-theory spectrum of the Waldhausen category  $\mathbf{FG}(M)$ . The homotopy groups  $\pi_i(\mathbf{G}(M)) := G_i(M)$  are defined to be the G-theory groups of  $M$ .*

**Remark 2.3.** *The G-theory of a Noetherian ring is defined in terms of the exact category of finitely generated modules, where the exact sequences are not necessarily split. One can also consider a version of G-theory with only split exact sequences of finitely generated modules and in the literature this is sometimes referred to as the  $G^\oplus$ -theory. Strictly speaking, our G-theory of  $\mathbb{F}_1$ -algebras is an analogue of this  $G^\oplus$ -theory. In the sequel we are mostly going to consider situations, where this distinction is immaterial.*

**Example 2.4.** *The category  $\mathbf{FG}(\mathbb{F}_1)$  is nothing but the category of finite pointed sets. It is a Waldhausen category with isomorphisms (pointed set bijections) as weak equivalences.*

The cofibrations are simply based injections as any injective pointed set map is split, i.e.,  $\iota : S_1 \hookrightarrow S_2$  is split by  $\theta : S_2 \rightarrow S_1$  sending  $s \mapsto \iota^{-1}(s)$  if  $s \in \text{Im}(\iota)$ , otherwise  $s \mapsto *_{S_1}$ . It is known that the Waldhausen K-theory spectrum of  $\mathbf{FG}(\mathbb{F}_1)$  is homotopy equivalent to the sphere spectrum (see, e.g., [48]).

Let  $S_1 \hookrightarrow S_2 \xrightarrow{p} S_2/S_1$  be a cofibration sequence in  $\mathbf{FG}(M)$ . By definition there is a splitting  $M$ -module homomorphism  $S_2 \rightarrow S_1$ . We call such a cofibration sequence *split* if, in addition, there is an  $M$ -module homomorphism  $\sigma : S_2/S_1 \rightarrow S_2$ , such that  $p \circ \sigma = \text{id}_{S_2/S_1}$ , i.e.,  $S_2 \cong S_1 \vee S_2/S_1$ .

**Lemma 2.5.** *Let  $G$  be a group. In the Waldhausen category  $\mathbf{FG}(G_+)$  every cofibration sequence is split.*

*Proof.* Let  $S_1 \hookrightarrow S_2 \xrightarrow{p} S_2/S_1$  be a cofibration sequence in  $\mathbf{FG}(G_+)$ . Let us first observe that  $S_2/S_1$  can be identified with  $(S_2 \setminus S_1)_+$ . Since  $G$  is a group, the pointed set  $(S_2 \setminus S_1)_+$  is  $G_+$ -invariant, i.e., it is a finitely generated  $G_+$ -module. The splitting of  $p$  is now obtained by the basepoint preserving map  $\sigma : (S_2 \setminus S_1)_+ \rightarrow S_2$ , which sends every nonbasepoint element in  $(S_2 \setminus S_1)_+$  to itself in  $S_2$ . It is obvious that  $\sigma$  is a  $G_+$ -module homomorphism.  $\square$

Let us now address the issue of functoriality of this construction. Let us define the smash product of a right  $M$ -module  $S$  and a left  $M$ -module  $S'$  to be  $S \wedge_M S' := S \times S' / \{(sm, s') \sim (s, ms') \mid \forall m \in M, s \in S, s' \in S'\}$ . Given two  $\mathbb{F}_1$ -algebras  $M, N$ , an  $M - N$ -bimodule  $S$  is simultaneously a left  $M$ -module and a right  $N$ -module, such that  $m(sn) = (ms)n$  for all  $m \in M, s \in S$  and  $n \in N$ . If  $S'$  is an  $M - N$ -bimodule, then there is an  $N$ -action on  $S \wedge_M S'$  defined by  $(s, s')n = (s, s'n)$  for all  $n \in N$ . In the sequel, we denote  $S \wedge_{\mathbb{F}_1} S'$  simply by  $S \wedge S'$ . Given any  $\mathbb{F}_1$ -algebra homomorphism  $\alpha : M \rightarrow N$ , one can view  $N$  as an  $M - N$ -bimodule via the homomorphism  $\alpha : M \rightarrow N$  in an obvious manner. We define the base change functor  $\alpha_* : \mathbf{FG}(M) \rightarrow \mathbf{FG}(N)$  as  $S \mapsto S \wedge_M N$  and  $f \mapsto f \wedge \text{id}$  (on morphisms). Note that  $S \wedge_M N$  attains a canonical  $N$ -module structure, such that if  $S' \subset S$  is a finite set of the generators of  $S$  as an  $M$ -module, then  $S' \wedge \{1_N\}$  is a finite set of generators of  $S \wedge_M N$  as an  $N$ -module. Indeed, given any  $(s, n) \in S \wedge_M N$ , there are  $s' \in S'$  and  $m \in M$ , such that  $s = s'm$ . Now write  $(s, n)$  as  $(s'm, n) = (s', \alpha(m)n) = (s', 1_N)\alpha(m)n$ .

**Proposition 2.6.** *For any  $\mathbb{F}_1$ -algebra homomorphism  $\alpha : M \rightarrow N$ , the induced functor  $\alpha_* : \mathbf{FG}(M) \rightarrow \mathbf{FG}(N)$  is Waldhausen exact.*

*Proof.* The functor  $\alpha_*$  clearly sends the zero object to the zero object and preserves weak equivalences, which are simply isomorphisms in the respective categories. It also preserves cofibrations as they are split inclusions. It remains to check that pushouts along cofibrations are preserved. Let  $S_1 \coprod_S S_2$  denote the pushout of  $S_1 \xleftarrow{f_1} S \xrightarrow{f_2} S_2$  in  $\mathbf{FG}(M)$ , where the map  $S_1 \xleftarrow{f_1} S$  is a cofibration. Then there is a canonical map

$$(1) \quad \alpha_*(S_1) \coprod_{\alpha_*(S)} \alpha_*(S_2) \rightarrow \alpha_*(S_1 \coprod_S S_2),$$

induced by  $\alpha_*$  applied to the maps  $f_i : S_i \rightarrow S_1 \coprod_S S_2$ ,  $i = 1, 2$  and observing that  $\text{Im}[\alpha_*(S) \rightarrow \alpha_*(S_1 \coprod_S S_2)] = \star$ , where  $\star$  is the zero object in  $\alpha_*(S_1 \coprod_S S_2)$ .

It is clear that this canonical map is surjective. Now suppose  $(s, n) = (s', n') \in \alpha_*(S_1 \coprod_S S_2)$ . Then the equality holds for a combination of the following two reasons:

- (1)  $\exists x \in S$ , such that  $f_1(x) = s$  and  $f_2(x) = s'$  and  $n = n'$ ;
- (2)  $\exists$  factorization  $n' = mn$  in  $N$ , such that  $s = s'm$ .

In either case one can easily check that the preimages must agree.  $\square$

**Corollary 2.7.** *The association  $M \mapsto \mathbf{G}(M)$  is functorial with respect to  $\mathbb{F}_1$ -algebra homomorphisms. Much like the case of rings, the association  $M \mapsto \mathbf{FG}(M)$  is only pseudofunctorial, i.e., it respects the composition of morphisms only up to an isomorphism.*

**Remark 2.8.** *It is known that in the elementary approach the G-theory of Noetherian schemes is only functorial with respect to flat morphisms. If  $M$  is an  $\mathbb{F}_1$ -algebra, one may call an  $M$ -module  $S$  flat if the functor  $- \wedge_M S$  is exact, i.e., it commutes with all finite colimits and limits in the category of  $M$ -modules. One may also call an  $\mathbb{F}_1$ -algebra homomorphism  $\alpha : M \rightarrow N$  flat if  $N$  viewed as an  $M$ -module via  $\alpha$  is flat. The apparent ‘excess functoriality’ of G-theory is explained by Remark 2.3.*

**2.1. Transfer maps.** Suppose  $R \rightarrow S$  is a unital ring homomorphism such that  $S$  is a finitely generated and projective  $R$ -module. Then there are wrong-way *transfer maps*  $K_i(S) \rightarrow K_i(R)$  induced by the restriction of scalars. Its counterpart in the G-theory of noetherian schemes is a covariant functoriality with respect to proper maps (see 3.16.1 of [56]). There are similar *transfer maps* in the G-theory of  $\mathbb{F}_1$ -algebras. Let  $M \rightarrow N$  be an  $\mathbb{F}_1$ -algebra homomorphism, which makes  $N$  into a finitely generated  $M$ -module. Then the restriction of scalars functor  $\mathbf{FG}(N) \rightarrow \mathbf{FG}(M)$  is Waldhausen exact, whence it induces a map  $\mathbf{G}(N) \rightarrow \mathbf{G}(M)$ . For the readers’ convenience, we record this fact as

**Lemma 2.9.** *If  $M \rightarrow N$  is an  $\mathbb{F}_1$ -algebra homomorphism, such that  $N$  becomes a finitely generated  $M$ -module, then there is a transfer map  $\mathbf{G}(N) \rightarrow \mathbf{G}(M)$ .*

**Example 2.10.** *Let  $H \subset G$  be an inclusion of groups giving rise to an  $\mathbb{F}_1$ -algebra homomorphism  $H_+ \rightarrow G_+$ . Then  $G_+$  is finitely generated as an  $H_+$ -module if and only if the coset space  $G/H$  is finite, i.e.,  $H$  is a subgroup of finite index.*

**2.2. The multiplicative structure.** A strict symmetric monoidal structure  $- \otimes -$  on a Waldhausen category  $\mathcal{C}$  is called *bisect* if for all  $A, B \in \mathcal{C}$  the  $\mathcal{C}$ -endofunctors  $A \otimes -$  and  $- \otimes B$  are exact and for any pair of cofibrations  $A \rightarrow A', B \rightarrow B'$  the induced map  $A' \otimes B \coprod_{A \otimes B} A \otimes B' \rightarrow B \otimes B'$  is a cofibration. In [32] the authors constructed a symmetric monoidal category of *symmetric spectra*, whose homotopy category models the symmetric monoidal stable homotopy category. The Waldhausen K-theory of a Waldhausen category is a symmetric spectrum (valued in simplicial sets) in a canonical manner. Bisect functors induce a multiplication on the Waldhausen K-theory spectrum and, in fact, render it with the structure of a connective and *quasifibrant* (in particular, semistable) symmetric ring spectrum (see Proposition 6.1.1. of [23]). Recall that a symmetric spectrum  $X$  is called *quasifibrant* if  $X_n$  is a fibrant simplicial set and the adjoint to the structure map  $\tilde{\sigma} : X_n \rightarrow \Omega X_{n+1}$  is a weak homotopy equivalence for all  $n \geq 1$ . It is known that a map between quasifibrant symmetric spectra (more generally, between semistable symmetric spectra) is a weak equivalence if and only if it is a  $\pi_*$ -equivalence, which is not true for maps between arbitrary symmetric spectra.

**Lemma 2.11.** *Let  $G$  be a group. Then the symmetric monoidal bifunctor  $(S, S') \mapsto S \wedge S'$  with the diagonal  $G$ -action on the Waldhausen category  $\mathbf{FG}(G_+)$  is bisect.*



*Proof.* For any  $S, S' \in \mathbf{FG}(G_+)$ , the exactness of  $S \wedge -$  (and  $- \wedge S'$ ) is similar to the argument in the proof of Proposition 2.6. Suppose  $S_1 \twoheadrightarrow S$  and  $T_1 \twoheadrightarrow T$  are two cofibrations (necessarily split). There is a self-evident broken arrow in the commutative diagram below

$$\begin{array}{c}
 S \wedge T \longleftarrow S \wedge T_1 \\
 \swarrow \quad \quad \quad \nwarrow \\
 S_1 \wedge T \quad \coprod_{S_1 \wedge T_1} \quad S \wedge T_1 \longleftarrow S \wedge T_1 \\
 \uparrow \quad \quad \quad \uparrow \\
 S_1 \wedge T \longleftarrow S_1 \wedge T_1
 \end{array}$$

Let  $S' = S/S_1$  and  $T' = T/T_1$ . It follows from Lemma 2.5 that

$$\begin{aligned}
 S_1 \wedge T \coprod_{S_1 \wedge T_1} S \wedge T_1 &\cong (S_1 \wedge T') \vee (S_1 \wedge T_1) \vee (S' \wedge T_1), \\
 S \wedge T &\cong (S_1 \wedge T') \vee (S_1 \wedge T_1) \vee (S' \wedge T_1) \vee (S' \wedge T').
 \end{aligned}$$

The broken arrow in the above diagram corresponds to the canonical inclusion of the first three wedge pieces. The splitting is obtained by sending the piece  $S' \wedge T'$  to the basepoint (identity otherwise). □

**Proposition 2.12.** *For a finite group  $G$ , the  $\mathbf{G}$ -theory spectrum  $\mathbf{G}(G_+)$  is canonically a symmetric ring spectrum. Furthermore,  $\oplus \pi_n(\mathbf{G}(G_+)) = \oplus \mathbf{G}_n(G_+)$  is a graded commutative ring with identity and  $\oplus \pi_{2n}(\mathbf{G}(G_+)) = \oplus \mathbf{G}_{2n}(G_+)$  is a commutative ring with identity.*

*Proof.* That  $\mathbf{G}(G_+)$  is a symmetric ring spectrum follows from the above Lemma 2.11 and Proposition 6.1.1. of [23]. The spectrum  $\mathbf{G}(G_+)$  is semistable because it is a connective and convergent spectrum (see Proposition 5.6.4. (2) of [32]). Now it follows from Proposition 6.25. of [48] that  $\oplus \pi_n(\mathbf{G}(G_+))$  is a graded ring. Since the pairing on  $\mathbf{FG}(G_+)$  is homotopy commutative, so is the ring spectrum  $\mathbf{G}(G_+)$ . It follows that the multiplication on  $\oplus \pi_n(\mathbf{G}(G_+))$  is graded commutative and on  $\oplus \pi_{2n}(\mathbf{G}(G_+))$  is commutative. □

Let  $G$  be a group and  $R$  be a commutative unital ring. Then there is a symmetric monoidal structure on the category of  $R[G]$ -modules, that are finitely generated and projective over  $R$ , given by  $- \otimes_R -$  with diagonal  $G$ -action. The symmetric monoidal structure of Lemma 2.11 above is similar to this one. Let  $\mathbf{FSet}$  denote the category of finite sets. Let  $G$  be any (unpointed) group. We denote by  $\mathbf{FSet}^G$  the category of finite sets with a  $G$ -action and  $G$ -maps. When  $G$  is a finite group, the K-theory of the symmetric monoidal category  $\mathbf{FSet}^G$  (under disjoint union) obtained by applying Segal's  $\Gamma$ -space machine is known to be (weakly) homotopy equivalent to the  $G$ -fixed point spectrum of the  $G$ -equivariant sphere spectrum, i.e.,  $\mathbf{K}(\mathbf{FSet}^G) \simeq \mathbb{S}^G$  (this result is presumably well-known and it is explicitly stated in Section 5 of [8]).

Let  $\mathbf{FSet}_*^G$  denote the category of finite pointed  $G$ -sets with pointed  $G$ -set maps. There is a functor  $P : \mathbf{FSet}^G \rightarrow \mathbf{FSet}_*^G$ , sending any finite  $G$ -set  $S$  to the pointed  $G$ -set  $S_+$  (adding a disjoint basepoint). The category  $\mathbf{FSet}^G$  (resp.  $\mathbf{FSet}_*^G$ ) is a symmetric monoidal category with respect to disjoint union (resp. pointed union) of  $G$ -sets (resp. pointed  $G$ -sets). The

functor  $P : \mathbf{FSet}^G \rightarrow \mathbf{FSet}_*^G$  is symmetric monoidal, i.e.,  $P(S_1 \amalg S_2) \cong P(S_1) \vee P(S_2)$ . There is a canonical functor  $P' : \mathbf{FSet}_*^G \rightarrow \mathbf{FSet}^G$  which simply sends a pointed finite  $G$ -set to the unpointed  $G$ -set after omitting the basepoint. This functor is also symmetric monoidal. Although the functors  $P$  and  $P'$  are not categorical equivalences, e.g.,  $P$  is not full, restricted to the category of isomorphisms they are symmetric monoidal equivalences, whence they have homotopy equivalent (connective) K-theory spectra obtained by the machinery of [55], for instance. By the May–Thomason uniqueness of infinite loop space machines [42], this construction will agree with that of Segal’s  $\Gamma$ -space machine. Therefore, we conclude  $\mathbf{K}(\mathbf{FSet}_*^G) \simeq \mathbb{S}^G$ . The functor  $P$  induces a morphism between the Grothendieck groups of the symmetric monoidal categories  $\mathbf{FSet}^G$  and  $\mathbf{FG}(G_+)$ . The Grothendieck group of  $\mathbf{FSet}^G$  carries a multiplication induced by the cartesian product of  $G$ -sets and this ring is the *Burnside ring*  $A(G)$ .

**Proposition 2.13.** *Let  $G$  be a finite group. Then there is an isomorphism  $A(G) \cong G_0(G_+)$  induced by  $P$ .*

*Proof.* Let us first verify that the map induced by  $P$  is an abelian group homomorphism  $A(G) \rightarrow G_0(G_+)$ . Let  $S' \subset S$  be an inclusion of  $G$ -sets, so that  $S = S' \amalg (S \setminus S')$  as  $G$ -sets and hence  $[S] = [S'] + [S \setminus S']$  in  $A(G)$ . Then  $P(S) = S_+ = S'_+ \vee (S \setminus S')_+$  and we need to verify that  $[S_+] = [S'_+] + [(S \setminus S')_+]$  in  $G_0(G_+)$ . The Waldhausen  $K_0$ -group of  $\mathbf{FG}(G_+)$  is the free abelian group generated by the isomorphism classes  $[S_+]$  of objects in  $\mathbf{FG}(G_+)$  modulo relations  $[S_+] = [S'_+] + [S_+/S'_+]$ , whenever  $S'_+ \rightarrow S_+$  is a cofibration. By Lemma 2.5 one can identify  $S_+/S'_+$  with  $(S \setminus S')_+$ , whence  $S'_+ \rightarrow S_+ \rightarrow (S \setminus S')_+$  is a cofibration sequence, i.e.,  $[S_+] = [S'_+] + [(S \setminus S')_+]$  in  $G_0(G_+)$ .

It is clear that  $P(S \times S') \cong P(S) \wedge P(S')$ , whence the induced map  $A(G) \rightarrow G_0(G_+)$  respects multiplication. Given a pointed  $G_+$ -set  $S_+$  there is a unique  $G$ -set  $S = S_+ \setminus \{\text{basepoint}\}$  such that  $P(S) = S_+$ . Note that it works even if  $S_+$  is singleton (containing only the basepoint) as the empty set  $\emptyset$  carries a unique  $G$ -set structure since  $\text{Aut}(\emptyset)$  is singleton.  $\square$

**Remark 2.14.** *This result agrees with the calculation in Example 5.1.2 of [9]. Extrapolating this result  $G_i(G_+)$ , for  $i \geq 1$ , may be regarded as the higher Burnside ring of  $G$ . Note that  $A(G)$  is freely generated by  $\{[G/H] \mid H \subset G \text{ subgroup}\}$  as an abelian group. It follows from Proposition 2.12 that  $G_i(G_+)$  is a module over the the Burnside ring  $A(G)$  for all  $i$ .*

Recall from Section 1.8 of [61] that a *category with sum and weak equivalences*  $\mathcal{C}$  is a category with sum  $- \vee -$ , i.e., categorical coproduct, and weak equivalences, such that if  $A_1 \rightarrow A'_1$  and  $A_2 \rightarrow A'_2$  are weak equivalences then so is  $A_1 \vee A_2 \rightarrow A'_1 \vee A'_2$ . Any cofibration category, i.e., weak equivalences = isomorphisms, admits categorical coproducts and by a forget of structure gives rise to a category with sum and weak equivalences (with a zero object). This is precisely the kind of input data to which Segal’s  $\Gamma$ -space machine can be applied to produce a spectrum [50].

**Proposition 2.15.** *Let  $\mathcal{C}$  be a cofibration category, viewed as a Waldhausen category. Suppose in addition that every cofibration is split, i.e., every cofibration sequence  $A \rightarrowtail B \twoheadrightarrow B/A$  is isomorphic to  $A \rightarrowtail A \vee B/A \twoheadrightarrow B/A$ . By a forget of structure we view  $\mathcal{C}$  as a category with sum and weak equivalences. Then the Segal machine applied to  $\mathcal{C}$  (viewed as a category with sum and weak equivalences) and the Waldhausen machine applied to  $\mathcal{C}$  (viewed as a Waldhausen category) produce weakly homotopy equivalent spectra.*

*Proof.* Let  $\mathbf{Cat}$  denote the category of small categories. Segal machine is obtained by applying the nerve construction producing a simplicial category  $N_\bullet \mathcal{C} : \Delta^{\text{op}} \rightarrow \mathbf{Cat}$ , such that  $N_n \mathcal{C} = \{(A_1, \dots, A_n, \text{choices})\}$  after a few identifications (see, e.g., Section 1.8 of [61]). In Waldhausen's notation there is a map of spectra induced by  $wN_\bullet \mathcal{C} \rightarrow w\mathcal{S}_\bullet \mathcal{C}$  sending

$$(A_1, \dots, A_n, \text{choices})$$

to

$$(\star \rightrightarrows A_1 \rightrightarrows A_1 \vee A_2 \rightrightarrows \dots \rightrightarrows A_1 \vee \dots \vee A_n, \text{ (fewer) choices}).$$

Now by our assumption every object of  $w\mathcal{S}_n \mathcal{C}$ , i.e., an  $n$ -step filtration diagram

$$(\star \rightrightarrows B_1 \rightrightarrows B_2 \rightrightarrows \dots \rightrightarrows B_n, \text{ choices})$$

is isomorphic to one of the form

$$(\star \rightrightarrows A_1 \rightrightarrows A_1 \vee A_2 \rightrightarrows \dots \rightrightarrows A_1 \vee \dots \vee A_n, \text{ choices}).$$

It follows that for every  $n$  that map  $wN_n \mathcal{C} \rightarrow w\mathcal{S}_n \mathcal{C}$  is an equivalence of categories, whence it induces a weak equivalence of simplicial sets (after applying the nerve). In other words, we have a map of bisimplicial sets, which is a levelwise weak equivalence. The assertion now follows from the Segal–Zisman Realization Lemma for bisimplicial sets (see, for instance, Lemma 5.1. of [59]). □

For any finite group  $G$ , the category  $\mathbf{FSet}_*^G$  is isomorphic to the category  $\mathbf{FG}(G_+)$ . Indeed, any finitely generated module over a finite  $\mathbb{F}_1$ -algebra  $G$  is necessarily finite and the zero element of  $G_+$  necessarily acts as the zero morphism of the module. In fact, by forgetting the cofibration structure of  $\mathbf{FG}(G_+)$  it can be regarded as a category with sum and weak equivalences as above. This is precisely the symmetric monoidal category  $\mathbf{FSet}_*^G$ . By Proposition 2.15 the map

$$wN_\bullet \mathbf{FG}(G_+) \rightarrow w\mathcal{S}_\bullet \mathbf{FG}(G_+)$$

induces a weak equivalence of spectra, since the cofibrations in  $\mathbf{FG}(G_+)$  are all split (see Lemma 2.5). Note that the Segal machine applied to  $\mathbf{FSet}_*^G$  produces  $\mathbb{S}^G$ , whence we conclude

**Theorem 2.16.** *For every finite group, there is a weak equivalence of spectra  $\mathbb{S}^G \rightarrow \mathbf{G}(G_+)$ , i.e.,  $\mathbf{G}(G_+)$  is a model of  $\mathbb{S}^G$ .*

**Corollary 2.17.** *It follows from the results of Segal–tom Dieck that for a finite group  $G$ ,*

$$\mathbf{G}(G_+) \simeq \bigvee_K BW_G(K)_+ \wedge \mathbb{S},$$

where  $K$  runs through representatives of conjugacy classes of subgroups of  $G$ . Here  $W_G(K)$  is the Weyl group  $N_G(K)/K$  with  $N_G(K)$  being the normalizer of  $K$  in  $G$ . It also follows from the Iriye–Nishida result that every torsion element in  $\oplus_n \mathbf{G}_n(G_+)$  is nilpotent.

**2.3. On the relation with  $\mathcal{A}$ -theory.** Let  $G$  be a finite group and  $BG$  be the simplicial classifying space. Let  $R(\star, G)$  denote the category of finite pointed  $G$ -simplicial sets, i.e., those which are free in a pointed sense and finitely generated over  $G$ . It turns out that  $R(\star, G)$  is a Waldhausen category with injective maps as cofibrations and weak homotopy equivalences as weak equivalences. It follows from Theorem 2.1.5. of [61] that the Waldhausen K-theory groups of  $R(\star, G)$  are isomorphic to  $\mathcal{A}_i(BG)$  (in fact,  $K_i(R(\star, G)) = \mathcal{A}_i(BG)$  can be taken as a definition).

**Proposition 2.18.** *Let  $G$  be a finite group. Then there is a commutative diagram of finitely generated abelian groups*

$$\begin{array}{ccc} G_i(G_+) & \xrightarrow{\quad} & \mathcal{A}_i(BG) \\ & \searrow [G] & \downarrow \\ & & G_i(G_+), \end{array}$$

where the diagonal arrow  $G_i(G_+) \rightarrow G_i(G_+)$  is multiplication by the element  $[G/\{0\}] = [G]$  in the Burnside ring  $A(G)$ .

*Proof.* Consider the exact functor

$$\begin{array}{ccc} \mathbf{FG}(G_+) & \rightarrow & R(\star, G) \\ S_+ & \mapsto & S_+ \wedge G_+, \end{array}$$

where  $S_+ \wedge G_+$  is viewed as a constant  $G$ -simplicial set with the  $G$ -action  $(s, h)g = (sg, hg)$ . There is another exact functor

$$\begin{array}{ccc} R(\star, G) & \rightarrow & \mathbf{FG}(G_+) \\ X & \mapsto & \pi_0(X), \end{array}$$

where  $\pi_0(X)$  is equipped with its induced  $G_+$ -module structure. The composition of the two exact functors produces the following commutative diagram in Wald

$$\begin{array}{ccc} \mathbf{FG}(G_+) & \xrightarrow{\quad} & R(\star, G) \\ & \searrow & \downarrow \\ & & \mathbf{FG}(G_+), \end{array}$$

where the diagonal arrow  $\mathbf{FG}(G_+) \rightarrow \mathbf{FG}(G_+)$  is the composition of the other two, which sends  $S_+ \mapsto S_+ \wedge G_+$ . Since the K-theory of  $R(\star, G)$  is isomorphic to the  $\mathcal{A}$ -theory of  $BG$ , applying the K-theory functor we get a commutative diagram

$$\begin{array}{ccc} G_i(G_+) & \xrightarrow{\quad} & \mathcal{A}_i(BG) \\ & \searrow [G] & \downarrow \\ & & G_i(G_+), \end{array}$$

where the diagonal homomorphism  $G_i(G_+) \rightarrow G_i(G_+)$  is multiplication by the element  $[G/\{0\}] = [G]$  in the Burnside ring  $A(G)$  (see Remark 2.14 above). Finally, it is known that  $\mathcal{A}_i(BG)$  is finitely generated for all  $i$  (see Theorem I of [5], also [21]).  $\square$

### 3. MACKEY AND GREEN STRUCTURE AND THE G-THEORETIC ASSEMBLY MAP

In this section, unless otherwise stated,  $G$  is a finite group and  $F$  is a field, such that the characteristic of  $F$  is coprime to the order of  $G$ . Let us comment further on the assumptions made. In the sequel, we frequently require that the canonical Cartan homomorphism from K-theory to G-theory be an isomorphism. This is true if the input ring is unital regular and Noetherian. Our assumptions above imply that  $F[G]$  is such a ring. However, it is an overkill for this purpose. We present a few general cases, where the group algebra is unital regular and Noetherian and the interested reader can try to adapt the machinery below to these cases. We also remark that the case of infinite groups is definitely very interesting, but necessarily more delicate.

**Example 3.1.** *Hall proved that  $F[G]$  is Noetherian, if  $G$  is polycyclic-by-finite and the characteristic of  $F$  is zero [29]. If  $G$ , in addition, is torsion free, then  $F[G]$  is (left) regular, (see Lemma 1 of [22]).*

**Example 3.2.** *If  $G$  is a finitely generated abelian group and  $R$  is a commutative Noetherian ring, such that  $qR = R$  for the order  $q$  of every torsion element in  $G$ , then  $\text{gldim } R[G] = \text{gldim } R + \text{rk}(G)$  (Theorem 1.7 of [44], also [3]). Therefore, if  $F$  is a field of characteristic zero (and  $\text{rk}(G) < \infty$ , e.g.,  $G$  finite), then once again using devissage one can conclude that  $\mathbf{G}(F[G]) \cong \mathbf{K}(F[G])$ .*

Let  $\mathbf{FSet}^G$  denote the category of finite right  $G$ -sets with  $G$ -maps. A monoidal abelian category  $(\mathcal{A}, \otimes)$ -valued pair of functors  $(M_*, M^*)$  on the category  $\mathbf{FSet}^G$  is called a *Mackey functor* if the following hold:

- (1)  $M_*$  is covariant and  $M^*$  is contravariant with  $M_*(S) = M^*(S)$  for any finite  $G$ -set  $S$ ,
- (2) For each pullback diagram in  $\mathbf{FSet}^G$

$$\begin{array}{ccc} U & \xrightarrow{F} & S \\ \downarrow H & & \downarrow h \\ T & \xrightarrow{f} & V \end{array}$$

one has  $F_*H^* = h^*f_*$ , where  $F_* = M_*(F)$ ,  $H^* = M^*(H)$  and so on,

- (3) The functor  $M^*$  sends finite coproducts to finite products, i.e., for any pair of  $G$ -sets  $S, T$ , the canonical inclusions of  $S$  and  $T$  into the disjoint union  $S \amalg T$  induces an isomorphism  $M^*(S \amalg T) \cong M^*(S) \oplus M^*(T)$ . In particular, it follows that  $M_*(\emptyset) = M^*(\emptyset) = 0$  and  $M_*$  sends finite coproducts to finite products.

For any arrow  $f$  in  $\mathbf{FSet}^G$  the induced map  $f_*$  (resp.  $f^*$ ) is known as the induction (resp. restriction) homomorphism. A *Green functor*  $M = (M_*, M^*)$  is a Mackey functor equipped with a pairing (natural transformation)  $M \otimes M \rightarrow M$  satisfying certain *Frobenius reciprocity* conditions, such that for any finite  $G$ -set  $S$  the abelian group  $M(S) = M_*(S) = M^*(S)$  becomes a commutative unital monoid object in  $\mathcal{A}$ . It is further required that the restriction homomorphisms respect the unital monoid structure. A Mackey functor  $M$  is a *module over*

a *Green functor*  $R$  if there is natural transformation  $R \otimes M \rightarrow M$  satisfying certain *Frobenius reciprocity* conditions, such that for any finite  $G$ -set  $S$  the induced map  $R(S) \otimes M(S) \rightarrow M(S)$  makes  $M(S)$  into a unital  $R(S)$ -module. For further details on Mackey and Green functors we refer the readers to [17, 35].

Let  $S$  be any finite  $G$ -set. Denote by  $\hat{S}$  the category whose objects are the elements of  $S$  and the morphisms are triples  $(s', g, s)$ , such that  $sg = s'$ . The composition is defined as  $(s'', h, s') \circ (s', g, s) = (s'', gh, s)$ .

**Lemma 3.3.** *Let  $G$  be a finite group. Then equipped with objectwise cofibrations and weak equivalences, the functor category  $[\hat{S}, \mathbf{FG}(\mathbb{F}_1)]$  (resp.  $[\hat{S}, \mathbf{FG}(F)]$ ) is again a Waldhausen category for any finite  $G$ -set  $S$ .*

*Proof.* This is an immediate consequence Theorem 4.2 of [35].  $\square$

Consequently, we may construct the Waldhausen K-theory of  $[\hat{S}, \mathbf{FG}(\mathbb{F}_1)]$  and  $[\hat{S}, \mathbf{FG}(F)]$ . The association  $S \mapsto [\hat{S}, \mathbf{FG}(\mathbb{F}_1)]$  (or  $S \mapsto [\hat{S}, \mathbf{FG}(F)]$ ) define a functor from  $\mathbf{FSet}^G \rightarrow \mathbf{Wald}$ .

**Proposition 3.4.** *Let  $G$  be a finite group and  $F$  be a field as above. Then the association  $S \mapsto K_0([\hat{S}, \mathbf{FG}(\mathbb{F}_1)])$  (resp.  $S \mapsto K_0([\hat{S}, \mathbf{FG}(F)])$ ) constitutes a Green functor on  $\mathbf{FSet}^G$ . Furthermore, the association  $S \mapsto K_n([\hat{S}, \mathbf{FG}(\mathbb{F}_1)])$  (resp  $S \mapsto K_n([\hat{S}, \mathbf{FG}(F)])$ ) constitutes a Mackey module functor over the aforementioned Green functor.*

*Proof.* The assertions concerning the association  $S \mapsto K_n([\hat{S}, \mathbf{FG}(F)])$  are proved in Theorem 1.4 and Theorem 1.6 of [19]. The assertions concerning the association  $S \mapsto K_n([\hat{S}, \mathbf{FG}(\mathbb{F}_1)])$  are proved in Theorem 5.1.3. of [35].  $\square$

There is a canonical Waldhausen exact functor  $\eta : \mathbf{FG}(\mathbb{F}_1) \rightarrow \mathbf{FG}(F)$ , which sends a finitely generated  $\mathbb{F}_1$ -module or a finite pointed set  $S_+$  to the  $F$ -module  $F[S]$ . A *morphism of Mackey (or Green) functors*  $M = (M_*, M^*) \rightarrow N = (N_*, N^*)$  is a natural transformation simultaneously between  $M_* \rightarrow N_*$  and  $M^* \rightarrow N^*$  (respecting all the extra structures). The following proposition can be easily verified:

**Proposition 3.5.** *For any finite group  $G$  and  $F$  as above, the functor  $\eta : \mathbf{FG}(\mathbb{F}_1) \rightarrow \mathbf{FG}(F)$  induces a morphism of Green functors  $K_0([- , \mathbf{FG}(\mathbb{F}_1)]) \rightarrow K_0([- , \mathbf{FG}(F)])$  and that of Mackey functors  $K_n([- , \mathbf{FG}(\mathbb{F}_1)]) \rightarrow K_n([- , \mathbf{FG}(F)])$  on  $\mathbf{FSet}^G$  for all  $n \geq 1$ .*

**3.1. Cartan assembly map.** We are now going to construct G-theoretic assembly maps. Let us first observe the following:

**Lemma 3.6.** *Let  $G$  be a finite group and  $H \subset G$  be any subgroup. For  $S = G/H$  there are exact equivalences between Waldhausen categories  $[\hat{S}, \mathbf{FG}(\mathbb{F}_1)] \cong \mathbf{FG}(H_+)$  and  $[\hat{S}, \mathbf{FG}(F)] \cong \mathbf{FG}(F[H])$ .*

*Proof.* The exact equivalence  $[\hat{S}, \mathbf{FG}(F)] \cong \mathbf{FG}(F[H])$  is proven in Theorem 3.2 of [19]. The proof for the exact equivalence  $[\hat{S}, \mathbf{FG}(\mathbb{F}_1)] \cong \mathbf{FG}(H_+)$  is similar and left to the reader.  $\square$

**Remark 3.7.** *Observe that for the  $G$ -set  $S = G/H$ , where  $H \subset G$  is any subgroup, we have*

$$K_n([\hat{S}, \mathbf{FG}(\mathbb{F}_1)]) = G_n(H_+) \text{ and } K_n([\hat{S}, \mathbf{FG}(F)]) \cong G_n(F[H]) \cong K_n(F[H]).$$

The *orbit category* of  $G$ , denoted by  $\text{Or}(G)$ , is defined to be the full subcategory of  $\mathbf{FSet}^G$  consisting of objects of the form  $G/H$ , where  $H \subset G$  is a subgroup. The association  $G/H = S \mapsto \mathbf{K}([\hat{S}, \mathbf{FG}(\mathbb{F}_1)])$  (resp.  $G/H = S \mapsto \mathbf{K}([\hat{S}, \mathbf{FG}(\mathbb{Q})])$ ) produces a covariant module spectrum  $\mathbf{K}_{G, \mathbb{F}_1}$  (resp.  $\mathbf{K}_{G, F}$ ) over the orbit category of  $G$  via the induction maps, i.e., a covariant functor from  $\text{Or}(G)$  to the category of (symmetric)  $\Omega$ -spectra. Thanks to the previous Lemma this covariant module spectrum over  $\text{Or}(G)$  has the property  $\pi_n(\mathbf{K}_{G, \mathbb{F}_1})(G/H) \cong G_n(H_+)$  (resp.  $\pi_n(\mathbf{K}_{G, F})(G/H) \cong G_n(F[H])$ ). By Lemma 4.4 of [14] we may now construct  $G$ -homology theories on the category of  $G$ -CW pairs by setting:

$$\begin{aligned} H_n^G(X, A; \mathbf{K}_{G, \mathbb{F}_1}) &= \pi_n(\text{map}_G(-, (X_+ \coprod_{A_+} \text{Cone}(A_+))) \wedge_{\text{Or}(G)} \mathbf{K}_{G, \mathbb{F}_1}(-)), \\ H_n^G(X, A; \mathbf{K}_{G, F}) &= \pi_n(\text{map}_G(-, (X_+ \coprod_{A_+} \text{Cone}(A_+))) \wedge_{\text{Or}(G)} \mathbf{K}_{G, F}(-)). \end{aligned}$$

Here  $-\wedge_{\text{Or}(G)}-$  denotes the *balanced smash product* between a pointed (contravariant)  $\text{Or}(G)$ -space and a (covariant)  $\text{Or}(G)$ -module spectrum, which produces a spectrum, in the usual sense. By definition, if  $X$  is a pointed contravariant  $\text{Or}(G)$ -space and  $Y$  is a covariant  $\text{Or}(G)$ -spectrum, the the balanced product is defined to be

$$X \wedge_{\text{Or}(G)} Y = \bigvee_{G/H \in \text{Or}(G)} (X(G/H) \wedge Y(G/H)) / \sim,$$

where  $\sim$  is the equivalence relation generated by  $(x\phi, y) \sim (x, \phi y)$  for all morphisms  $\phi : G/H \rightarrow G/K$  in  $\text{Or}(G)$  and points  $x \in X(G/K)$ ,  $y \in Y(G/H)$ .

**Remark 3.8.** *These  $G$ -homology theories may not possess the desirable induction structure; nevertheless, the assembly maps make sense in any  $G$ -homology theory.*

Now the  $G$ -projection  $EG_+ \rightarrow (G/G)_+ = \text{pt}_+$  produces assembly maps

$$H_n(BG; \mathbb{S}) \cong H_n^G(EG; \mathbf{K}_{G, \mathbb{F}_1}) \rightarrow H_n^G(\text{pt}; \mathbf{K}_{G, \mathbb{F}_1}) \cong G_n(G_+)$$

and

$$H_n(BG; \mathbf{K}_F) \cong H_n^G(EG; \mathbf{K}_{G, F}) \rightarrow H_n^G(\text{pt}; \mathbf{K}_{G, F}) \cong K_n(F[G]).$$

**Theorem 3.9.** *For any finite group  $G$  and a field  $F$ , such that the characteristic of  $F$  is coprime to the order of  $G$ , there is a homotopy commutative diagram of spectra*

$$\begin{array}{ccc} BG_+ \wedge \mathbb{S} & \longrightarrow & BG_+ \wedge \mathbf{K}_F \\ \downarrow & & \downarrow \\ \mathbf{G}(G_+) & \longrightarrow & \mathbf{K}(F[G]), \end{array}$$

where the vertical maps are the assembly maps.

*Proof.* Under the assumptions  $F[G]$  is a regular Noetherian ring, whence  $\mathbf{G}(F[G]) \simeq \mathbf{K}(F[G])$ . Thanks to Proposition 3.5 the exact functor  $\eta : \mathbf{FG}(\mathbb{F}_1) \rightarrow \mathbf{FG}(F)$  induces a map of  $\text{Or}(G)$ -module spectra  $\mathbf{K}_{G, \mathbb{F}_1} \rightarrow \mathbf{K}_{G, F}$ . It follows that there is a natural transformation between the  $G$ -homology theories defined by these spectra, induced by the following homotopy commutative diagram of spectra:

$$\begin{array}{ccc}
\mathrm{map}_G(-, EG_+) \wedge_{\mathrm{Or}(G)} \mathbf{K}_{G, \mathbb{F}_1}(-) & \longrightarrow & \mathrm{map}_G(-, EG_+) \wedge_{\mathrm{Or}(G)} \mathbf{K}_{G, F}(-) \\
\downarrow & & \downarrow \\
\mathrm{map}_G(-, (G/G)_+) \wedge_{\mathrm{Or}(G)} \mathbf{K}_{G, \mathbb{F}_1}(-) & \longrightarrow & \mathrm{map}_G(-, (G/G)_+) \wedge_{\mathrm{Or}(G)} \mathbf{K}_{G, F}(-)
\end{array}$$

This diagram of balanced product spectra reduces to the diagram in our assertion.  $\square$

**Remark 3.10.** In [10] the authors developed K-theory for (pointed) monoids and showed that the  $K_i(G_+) \cong \pi_i(BG_+ \wedge \mathbb{S})$  (see Corollary 4.3. of *ibid.*), where  $G$  is a finite group. Hence the assembly map  $BG_+ \wedge \mathbb{S} \rightarrow \mathbf{G}(G_+)$  is a model for the Cartan homomorphism from K-theory to G-theory. The dual  $G$ -cohomological assembly map will be related to the completion map in stable cohomotopy of Segal conjecture.

The Cartan assembly map  $K_i(G_+) \rightarrow G_i(G_+)$  will not be an isomorphism. In [36], Loday constructed an assembly map  $BG_+ \wedge \mathbf{K}_{\mathbb{Z}} \rightarrow \mathbf{K}_{\mathbb{Z}[G]}$  and defined the homotopy cofibre to be the *Whitehead spectrum*  $\mathbf{Wh}(G, \mathbb{Z})$  of  $G$  over  $\mathbb{Z}$ . The homotopy groups of this spectrum are called the (higher) *Whitehead groups* of  $G$  over  $\mathbb{Z}$ . Analogously, the Whitehead spectrum of  $G$  over  $F$ , such that the order of  $G$  is invertible in  $F$ , is defined to be  $\mathbf{Wh}(G, F) := \mathrm{hocofib}[BG_+ \wedge \mathbf{K}_F \rightarrow \mathbf{K}_{F[G]}]$ . Motivated by this construction, we define

**Definition 3.11.** For a finite group  $G$ , the (G-theoretic) Loday–Whitehead spectrum of  $G$  over  $\mathbb{F}_1$ , denoted by  $\mathbf{Wh}(G, \mathbb{F}_1)$ , is defined to be the homotopy cofibre of the Cartan assembly map

$$\mathrm{hocofib}[BG_+ \wedge \mathbb{S} \rightarrow \mathbf{G}(G_+)].$$

We refer to its homotopy groups, denoted by  $\mathrm{Wh}_n(G, \mathbb{F}_1)$ , as the Loday–Whitehead groups of  $G$  over  $\mathbb{F}_1$ , which measure the deviation of the Cartan assembly map from being an isomorphism.

**Remark 3.12.** We have already observed that  $\mathbf{G}(\mathbb{F}_1) \simeq \mathbb{S}$ . Rather suggestively, one may write the above assembly map as  $BG_+ \wedge \mathbf{G}_{\mathbb{F}_1} \rightarrow \mathbf{G}_{\mathbb{F}_1[G]}$ , where  $\mathbb{F}_1[G] = G_+$ .

**Lemma 3.13.** For a finite group  $G$ , the groups  $\mathrm{Wh}_n(G, \mathbb{F}_1)$  are all finitely generated (and the higher ones are finite).

*Proof.* The assertion follows from Theorem 2.16 and the known results about the finiteness of stable homotopy of  $BG$ , obtained by a spectral sequence argument.  $\square$

Let  $\mathbf{C}(G, F)$  denote the homotopy cofibre of the map  $\mathbf{G}(G_+) \rightarrow \mathbf{K}(F[G])$  in the above Theorem 3.9. We denote its homotopy groups  $\pi_i(\mathbf{C}(G, F)) = C_i(G, F)$ .

**Proposition 3.14.** Let  $G$  be a finite group and  $F$  be a finite field, whose characteristic does not divide the order of  $G$ . Then, for all  $i \geq 2$ , there is an exact sequence

$$0 \rightarrow C_{2i}(G, F) \rightarrow G_{2i-1}(G_+) \rightarrow K_{2i-1}(F[G]) \rightarrow C_{2i-1}(G, F) \rightarrow G_{2i-2}(G_+) \rightarrow 0.$$

*Proof.* Due to the assumptions on  $F$  and  $G$ , the group algebra  $F[G]$  is a finite dimensional semisimple  $F$ -algebra (Maschke’s Theorem). Now by Wedderburn’s Theorem and the triviality of the Brauer group of finite fields, we get  $F[G] \simeq \prod_{k=1}^l M_{n_k}(F)$ . Since algebraic K-theory respects finite products and it is Morita invariant, one has  $K_i(F[G]) \simeq \prod_{k=1}^l K_i(F)$ . From



Quillen's computation, it is known that  $K_{2i}(F) = \{0\}$  for any finite field  $F$  [45]. The assertion now follows by inserting these trivial groups in the long exact sequence of homotopy groups arising from the homotopy fibration

$$\mathbf{G}(G_+) \rightarrow \mathbf{K}(F[G]) \rightarrow \mathbf{C}(G, F).$$

□

**Remark 3.15.** *At the tail-end of the long exact sequence one finds*

$$0 \rightarrow C_2(G, F) \rightarrow G_1(G_+) \rightarrow K_1(F[G]) \rightarrow C_1(G, F) \rightarrow G_0(G_+) \rightarrow \mathbb{Z}^l \rightarrow C_0(G, F).$$

*It is also known that  $K_{2i-1}(\mathbb{F}_q) \simeq \mathbb{Z}/(q^i - 1)$  for all  $i \geq 1$ . However, in order to extract information about  $G_i(G_+)$  from the above sequence one needs a bit more information, which we are unable to provide at the moment.*

**Proposition 3.16.** *Let  $G$  be a finite group and  $F$  be a finite field, whose characteristic does not divide the order of  $G$ . Then, for all  $i \geq 2$ , there is an exact sequence*

$$0 \rightarrow \mathrm{Wh}_{2i}(G, F) \rightarrow H_{2i-1}(BG; \mathbf{K}_F) \rightarrow K_{2i-1}(F[G]) \rightarrow \mathrm{Wh}_{2i-1}(G, F) \rightarrow H_{2i-2}(BG; \mathbf{K}_F) \rightarrow 0.$$

*Proof.* It follows from the vanishing of  $K_{2i}(F[G])$  for  $i \geq 1$  as argued above (see the proof of Proposition 3.14) and from the fact that  $BG_+ \wedge \mathbf{K}_F \rightarrow \mathbf{K}_{F[G]} \rightarrow \mathbf{Wh}(G, F)$  is a homotopy fibration (by construction). □

**Remark 3.17.** *Once again, at the tail-end of the long exact sequence one finds*

$$0 \rightarrow \mathrm{Wh}_2(G, F) \rightarrow H_1(BG; \mathbf{K}_F) \rightarrow K_1(F[G]) \rightarrow \mathrm{Wh}_1(G, F) \rightarrow H_0(G; \mathbb{Z}) \rightarrow \mathbb{Z}^l \rightarrow \mathrm{Wh}_0(G, F).$$

Since  $H_*(BG; \mathbb{S})$  (resp.  $H_*(BG; \mathbf{K}_F)$ ) is the value of a generalized homology theory on  $BG_+$ , it is computable by the first quadrant homological Atiyah–Hirzebruch spectral sequence, whose  $E_{r,s}^2$ -terms looks like  $H_r(BG; \pi_s(\mathbb{S}))$  (resp.  $H_r(BG; K_s(F))$ ). The K-theoretic spectral sequence  $H_r(BG; K_s(F))$  is particularly accessible to computation, since the coefficients  $\pi_s(\mathbf{K}_F) = K_s(F)$  are completely known.

#### 4. COMBINATORIAL GRAYSON OPERATIONS ON G-THEORY

The constructions in this section are motivated by the well-known  $\lambda$ -operations on K-theory. For the benefit of the reader we briefly sketch the construction of the  $\lambda$ -structure on higher K-theory of a commutative unital ring (following Kratzer–Quillen).

**4.1.  $\lambda$ -operations on higher K-theory.** For every natural number  $k$ ,  $\lambda^k$  is an operation on the higher algebraic K-theory, which is nonadditive on  $K_0$ . Recall that a commutative unital ring  $R$  is called a *pre- $\lambda$ -ring* if it is equipped with operations  $\{\lambda^k\}_{k \in \mathbb{N}}$  satisfying:

- $\lambda^0(x) = 1$  and  $\lambda^1(x) = x$ ,
- $\lambda^k(x + y) = \sum_{i=0}^k \lambda^i(x) \lambda^{k-i}(y)$ .

If the  $\lambda$ -operations on a pre- $\lambda$ -ring  $R$  satisfy, in addition, the following conditions:

- $\lambda^k(1) = 0$  for all  $k \geq 2$ ,
- $\lambda^k(xy) = P_k(\lambda^1(x), \dots, \lambda^k(x), \lambda^1(y), \dots, \lambda^k(y))$ ,
- $\lambda^k(\lambda^l(x)) = P_{k,l}(\lambda^1(x), \dots, \lambda^{kl}(x))$

then it is called a  $\lambda$ -ring. Here  $P_k$  and  $P_{k,l}$  are universal polynomials with integral coefficients, which are intimately related to symmetric functions [2]. A ring homomorphism between two  $\lambda$ -rings, which commutes with all the  $\lambda$ -operations is called a  $\lambda$ -homomorphism. Let  $G$  be any group and  $A$  be any commutative ring with identity. The  $\lambda$ -operations on higher K-theory of  $A$  are constructed by the following steps:

- (1) Let  $R_G(A)$  denote the Grothendieck group of the exact category of  $G$ -representation on finitely generated and projective  $A$ -modules. It attains a commutative unital ring structure (Swan ring) via the tensor product of representations with diagonal  $G$ -action. In [54] Swan showed that the maps

$$\begin{aligned}\lambda^k : R_G(A) &\rightarrow R_G(A) \\ [V] &\mapsto [\wedge^k V]\end{aligned}$$

define a  $\lambda$ -ring structure on  $R_G(A)$ .

- (2) For any  $G$ -representation on a finitely generated and projective  $A$ -module  $V$ , one has a homomorphism  $G \rightarrow GL(A)$ , which is unique up to conjugation. This gives rise to a continuous map  $r(V) : BG \rightarrow BGL(A)^+$ , which produces a well-defined homomorphism  $r : R_G(A) \rightarrow [BG, BGL(A)^+]$ .
- (3) Let  $X$  be any connected pointed finite CW complex. Then  $[X, BGL(A)^+]$  admits a commutative ring structure coming from the  $H$ -group structure on  $BGL(A)^+$  [37]. Using the homomorphism  $r$  of the previous step one constructs maps

$$\lambda^k : BGL(A)^+ \rightarrow BGL(A)^+,$$

which are well-defined up to homotopy (cf. Section 5 of [34]).

- (4) Let us set

$$\begin{aligned}\lambda^k : [X, BGL(A)^+] &\rightarrow [X, BGL(A)^+] \\ [g] &\mapsto [\lambda^k \circ g]\end{aligned}$$

Owing to the previous step these are well-defined set maps.

- (5) The Kratzer–Quillen Theorem says that, equipped with the above structures,  $K_0(A) \times [X, BGL(A)^+]$  becomes a  $\lambda$ -ring [34, 30]. It is canonically a  $K_0(A)$ -augmented  $\lambda$ -ring, i.e., the projection  $K_0(A) \times [X, BGL(A)^+] \rightarrow K_0(A)$  is a  $\lambda$ -homomorphism.
- (6) Setting  $X = S^n$  for  $n \geq 1$ , we obtain the  $\lambda$ -structure on higher K-theory. Since for  $n \geq 1$   $S^n$  is a co- $H$ -group the product structure on  $K_n(A) = [S^n, BGL(A)^+]$  is trivial (see Lemma 5.2 of [34]). It follows that for all  $n \geq 1$  the maps  $\lambda^k : K_n(A) \rightarrow K_n(A)$  are group homomorphisms; however, they are not group endomorphisms on  $K_0(A)$ .

**4.2. Combinatorial Grayson operations.** Let  $\mathcal{C}$  denote the Quillen exact category of finitely generated and projective modules over a commutative and unital ring. One could try to define the  $\lambda$ -operations directly on the  $\mathcal{S}_\bullet$ -construction  $\lambda^k : \mathcal{S}_\bullet \mathcal{C} \rightarrow \mathcal{S}_\bullet \mathcal{C}$ . Since  $K_i(\mathcal{C}) := \pi_{i+1}(|w\mathcal{S}_\bullet \mathcal{C}|)$ , such operations would necessarily be additive; however, we observed above that the  $\lambda$ -operations on  $K_0$  are not additive. Grayson overcame this difficulty in [26] by using an *undelooped* model for K-theory and defining certain combinatorial operations therein. Gunnarsson–Schwänzl extended Grayson’s construction to Waldhausen categories satisfying some further hypotheses in [27]. We construct operations on G-theory using the results in *ibid.*. Let us remark that the constructions in *ibid.* are much more general as the authors

were motivated by the construction of the total Segal operation on Waldhausen  $A$ -theory; we only focus on the cases that are relevant for our purposes, where many simplifications occur. Recall that a cofibration category is a Waldhausen category, whose weak equivalences are isomorphisms. A cofibration category  $\mathcal{C}$  is said to have the *extension property*, if whenever there is a commutative diagrams of (horizontal) cofibrations sequences in  $\mathcal{C}$

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow p & & \downarrow i & & \downarrow q \\ A' & \longrightarrow & B' & \longrightarrow & C' \end{array}$$

with extremal vertical arrows being cofibrations as indicated, it follows that  $i$  is also a cofibration. We claim

**Lemma 4.1.** *Let  $G$  be a group, so that we may view  $G_+$  as an  $\mathbb{F}_1$ -domain. Then as a cofibration category  $\mathbf{FG}(G_+)$  has the extension property.*

*Proof.* Suppose that we are given a commutative diagram of cofibration sequences in  $\mathbf{FG}(G_+)$

$$\begin{array}{ccccc} M & \longrightarrow & N & \longrightarrow & P \\ \downarrow p & & \downarrow i & & \downarrow q \\ M' & \longrightarrow & N' & \longrightarrow & P' \end{array}$$

with extremal vertical arrows as cofibrations. Since cofibrations in  $\mathbf{FG}(G)$  are split monomorphisms, using the fact that  $G$  is a group, one may write identify  $N$  (resp.  $N'$ ) with  $M \vee P$  (resp.  $M' \vee P'$ ) as  $G_+$ -modules. Suppose that  $p, q$  are split by  $s, t$ . Then  $s \vee t : N' \cong M' \vee P' \rightarrow M \vee P \cong N$  is a splitting of  $i$ , proving that it is a cofibration in  $\mathbf{FG}(G)$ .  $\square$

**Remark 4.2.** *Let  $R$  be a Noetherian unital ring, so that  $\mathbf{FG}(R)$  is an abelian category. We may regard  $\mathbf{FG}(R)$  as a cofibration category, whose cofibrations are simply monomorphisms. Then  $\mathbf{FG}(R)$  has extension property. Indeed, given any morphism between cofibration sequences (with extremal vertical cofibrations)*

$$\begin{array}{ccccc} M & \longrightarrow & N & \longrightarrow & P \\ \downarrow & & \downarrow i & & \downarrow \\ M' & \longrightarrow & N' & \longrightarrow & P' \end{array}$$

*simply use the Snake Lemma to deduce that  $i$  is a monomorphism.*

Consider the symmetric monoidal cofibration category  $(\mathcal{C}, \wedge)$ , where  $\mathcal{C}$  is the category of finitely generated modules  $\mathbf{FG}(G_+)$  over an  $\mathbb{F}_1$ -algebra  $G_+$  ( $G$  being a group). For  $S, T \in \mathbf{FG}(G_+)$ , we equip  $S \wedge T$  with the diagonal  $G_+$ -action. We know that in this case  $- \wedge -$  is biexact and the cofibration category satisfies the extension property. Then we have the following two operations:

- (1) An analogue of the tensor product of modules

$$\begin{aligned} \boxtimes : \mathcal{C} \times \mathcal{C} &\rightarrow \mathcal{C} \\ (S, T) &\mapsto S \wedge T \end{aligned}$$

with diagonal  $G_+$ -action.

- (2) An analogue of the (generalized) exterior product of modules; let  $F_k(\mathcal{C})$  denote the category of  $k$ -filtered objects of  $\mathcal{C}$ , i.e., one whose objects are cofibration strings  $\underline{S} := S_1 \rightarrowtail S_2 \rightarrowtail \cdots \rightarrowtail S_k$ . Then consider

$$\begin{aligned} \diamond^k : F_k(\mathcal{C}) &\rightarrow \mathcal{C} \\ \underline{S} &\mapsto \diamond^k(\underline{S}) \end{aligned}$$

Now we describe the diamond functor  $\diamond^k$  rather explicitly. First consider the  $k$ -th diamond product  $\diamond^k S := S \diamond \cdots \diamond S$  ( $k$  times) of a single module  $S$ . Consider the  $G_+$ -submodule  $Q$  of  $\boxtimes^k S$  generated by tuples  $(s_1, \dots, s_k)$ , such that  $s_i = s_j$  for some pair  $i, j$  and  $i \neq j$ . We define  $\diamond^k S := \boxtimes^k S / Q$ , i.e., there is a cofibration sequence  $Q \rightarrowtail \boxtimes^k S \rightarrow \diamond^k S$ . Note that  $Q \rightarrowtail \boxtimes^k S_k$  is split because  $G$  is a group, whence the map is indeed a cofibration in  $\mathbf{FG}(G_+)$ . Now given  $\underline{S} := S_1 \rightarrowtail S_2 \rightarrowtail \cdots \rightarrowtail S_k$ , there is a canonical map  $S_1 \boxtimes \cdots \boxtimes S_k \rightarrow \boxtimes^k S_k$ . Now we define  $\diamond^k(\underline{S})$  to be the  $G_+$ -submodule of  $\diamond^k S_k$  generated by the image of the composite

$$S_1 \boxtimes \cdots \boxtimes S_k \rightarrow \boxtimes^k S_k \rightarrow \diamond^k S_k.$$

**Remark 4.3.** For any  $S_+ = S \coprod \{\star\} \in \mathbf{FG}(G_+)$ , using Lemma 2.5 one can ascertain that  $\diamond^k S_+$  is the set of ordered subsets of  $S$  (without repetition) of cardinality  $k$  along with the disjoint basepoint  $\star$ . It carries a canonical pointed  $G_+$ -action. In particular; if  $S$  has cardinality  $n$ , then it is clear that  $\diamond^k S = \star$  for all  $k > n$ .

In order to construct operations on G-theory from the above data, one needs further preparation. It turns out that the  $\mathcal{S}_\bullet$ -construction is not suitable and one needs a variant called the  $\mathcal{G}_\bullet$ -construction (see [25]). In order to avoid notational confusion due to the abundance of  $G$ 's appearing in various forms, we denote this construction by  $\mathfrak{G}_\bullet$ . Much like the  $\mathcal{S}_\bullet$ -construction, for any  $\mathcal{C} \in \mathbf{Wald}$  the  $\mathfrak{G}_\bullet$ -construction is a functor  $\mathbf{Wald} \rightarrow \mathbf{Wald}^{\Delta^{\text{op}}}$  defined by the cartesian square

$$\begin{array}{ccc} \mathfrak{G}_\bullet \mathcal{C} & \longrightarrow & PS_\bullet \mathcal{C} \\ \downarrow & & \downarrow \\ PS_\bullet \mathcal{C} & \longrightarrow & \mathcal{S}_\bullet \mathcal{C}, \end{array}$$

where  $PS_\bullet \mathcal{C}$  is the path object  $PS_n \mathcal{C} = \mathcal{S}_{n+1} \mathcal{C}$  and  $PS_\bullet \mathcal{C} \rightarrow \mathcal{S}_\bullet \mathcal{C}$  is given by the boundary map  $d_0 : \mathcal{S}_{n+1} \mathcal{C} \rightarrow \mathcal{S}_n \mathcal{C}$ . Since  $PS_\bullet \mathcal{C}$  is simplicially homotopic to a point, there is a canonical map  $|w\mathfrak{G}_\bullet \mathcal{C}| \rightarrow \Omega |w\mathcal{S}_\bullet \mathcal{C}|$  which is a weak equivalence if  $\mathcal{C}$  is *pseudo-additive* (see Theorem 2.6 of [28]).

**Remark 4.4.** We do not reproduce the exact definition of pseudo-additivity. It suffices to say that all Quillen exact categories are pseudo-additive (see Remark 2.7 of *ibid.*) and so are cofibration categories  $\mathcal{C}$ , whose cofibration sequences are split in  $\mathcal{C}$  (see Remark 2.4 (3) of *ibid.*). Therefore, for any group  $G$  and any Noetherian unital ring  $R$  the cofibration categories  $\mathbf{FG}(G_+)$  and  $\mathbf{FG}(R)$  are both pseudo-additive. Consequently, we have

$$\mathbf{G}(G_+) \cong |w\mathfrak{G}_\bullet \mathbf{FG}(G_+)| \text{ and } \mathbf{G}(R) \cong |w\mathfrak{G}_\bullet \mathbf{FG}(R)|.$$

Grayson enlisted five conditions  $((E1), \dots, (E5))$ , which  $\boxtimes$  and  $\diamond^k$  must satisfy so that one can construction the desired operations [26] (see also [27]).

**Lemma 4.5.** *For any group  $G$ , the operations  $\boxtimes$  and  $\diamond^k$  defined on  $\mathbf{FG}(G_+)$  above satisfy the conditions  $(E1), \dots, (E5)$ .*

Let  $\mathcal{C}$  be a cofibration category. The maps  $\mathcal{C} \cong (P\mathcal{S}_\bullet\mathcal{C})_0 \rightarrow P\mathcal{S}_\bullet\mathcal{C}$  and the zero map  $\mathcal{C} \rightarrow P\mathcal{S}_\bullet\mathcal{C}$  compose with  $d_0$  to the same image inside  $\mathcal{S}_\bullet\mathcal{C}$ . Therefore, the pullback definition of  $\mathfrak{S}_\bullet\mathcal{C}$  produces a map  $\mathcal{C} \rightarrow \mathfrak{S}_\bullet\mathcal{C}$ , which can be iterated to produce maps  $\mathfrak{S}^n\mathcal{C} \rightarrow \mathfrak{S}^{n+1}\mathcal{C}$  for all  $n \in \mathbb{N}$ . Let  $g\mathcal{C}$  be the simplicial set obtained by setting  $g_m\mathcal{C} = \text{Obj}\mathfrak{S}_m\mathcal{C}$ , i.e., extracting the object sets from the simplicial category  $\mathfrak{S}_\bullet\mathcal{C}$ . Then there is a weak equivalence  $|g\mathcal{C}| \xrightarrow{\sim} |w\mathfrak{S}_\bullet\mathcal{C}|$  if the weak equivalences in  $\mathfrak{S}_\bullet\mathcal{C}$  are all isomorphisms (see Lemma 2.14 of [28]).

For any Quillen exact category  $\mathcal{C}$  equipped with two functors  $\boxtimes$  and  $\diamond^k$  satisfying the conditions  $(E1), \dots, (E5)$ , there are simplicial Grayson maps

$$\omega^k : \text{sub}_k w\mathfrak{S}_\bullet\mathcal{C} \rightarrow w\mathfrak{S}_\bullet^k\mathcal{C},$$

where  $\text{sub}_k$  denotes the  $k$ -fold edgewise subdivision functor. The construction of the maps  $\omega^k$  are quite involved and we refer the readers to the original reference [26] (or Section 2 of [27]). Passing to the homotopy groups of the geometric realization one obtains the operations on the K-theory groups of  $\mathcal{C}$ . Let  $G$  be a group. Setting  $\mathcal{M}_n = \mathbf{FG}(G_+)$  for all  $n \geq 0$  in Theorem 4.1 of [27] we obtain

**Proposition 4.6.** *There are simplicial Grayson maps*

$$\omega^k : \text{sub}_k w\mathfrak{S}_\bullet\mathbf{FG}(G_+) \rightarrow w\mathfrak{S}_\bullet^k\mathbf{FG}(G_+).$$

**Theorem 4.7.** *There are well-defined operations on the  $G$ -theory of an  $\mathbb{F}_1$ -algebra  $G_+$  ( $G$  being a group)*

$$\omega^k : G_i(G_+) \rightarrow G_i(G_+)$$

for all  $i \geq 0$ , which are induced by Grayson's maps.

*Proof.* It is well known that the canonical map  $\text{sub}_k w\mathfrak{S}_\bullet\mathbf{FG}(G_+) \rightarrow w\mathfrak{S}_\bullet\mathbf{FG}(G_+)$  is a weak equivalence after geometric realization. The assertion is more obvious after identifying  $|g\mathbf{FG}(G_+)| \xrightarrow{\sim} |w\mathfrak{S}_\bullet\mathbf{FG}(G_+)|$ . Theorem 2.8 of [28] gives us a weak equivalence  $|w\mathfrak{S}_\bullet\mathbf{FG}(G_+)| \xrightarrow{\sim} \Omega|w\mathcal{S}_\bullet\mathbf{FG}(G_+)| = \mathbf{G}(G_+)$ , since  $\mathbf{FG}(G_+)$  is a pseudo-additive cofibration category (see Remark 4.4 above). It follows from Proposition 1.55' of *ibid.* that there is a weak equivalence  $|w\mathfrak{S}_\bullet\mathcal{C}| \xrightarrow{\sim} |w\mathfrak{S}_\bullet^k\mathcal{C}|$  whenever  $\mathcal{C}$  is a pseudo-additive cofibration category (it is explicitly observed on Page 264 of *ibid.*). Therefore, taking the geometric realization of the map

$$\omega^k : \text{sub}_k w\mathfrak{S}_\bullet\mathbf{FG}(G_+) \rightarrow w\mathfrak{S}_\bullet^k\mathbf{FG}(G_+)$$

and passing to homotopy groups we get the desired operations.  $\square$

**Remark 4.8.** *Let  $\mathcal{P}(R, G)$  be the category of representations of a discrete group  $G$  over finitely generated and projective  $R$ -modules, where  $R$  is a commutative unital ring. Let  $M \boxtimes N$  stand for  $M \otimes_R N$  with diagonal  $G$ -action for any  $M, N \in \mathcal{P}(R, G)$ . Let  $\underline{M} := M_1 \rightarrow M_2 \rightarrow \dots \rightarrow M_k$  be a string of monomorphisms of finitely generated and projective  $R$ -modules and let  $\diamond^k(\underline{M})$  be the image of  $M_1 \otimes_R \dots \otimes_R M_k$  in  $\wedge^k M_k$ , i.e., the  $k$ -th exterior power of  $M_k$  over  $R$ . Then  $\boxtimes$  and  $\diamond^k$  satisfy the conditions  $(E1), \dots, (E5)$  and Grayson used the associated maps*

$$\omega^k : \text{sub}_k w\mathfrak{S}_\bullet\mathcal{P}(R) \rightarrow w\mathfrak{S}_\bullet^k\mathcal{P}(R)$$

to construct operations on the higher algebraic K-theory of the exact category  $\mathcal{P}(R, G)$ . He also showed that these operations agree with the  $\lambda$ -operations on the higher algebraic K-theory of  $R$  (when  $G$  is trivial) as described above (see Subsection 4.1).

The combinatorics involved in the construction of the operations  $\omega^k$  are quite complicated. It is not straightforward to verify from the definition that endowed with these operations the higher algebraic K-theory attains a  $\lambda$ -structure. The problem is circumvented by identifying the operations with the already known  $\lambda$ -structure on higher algebraic K-theory. The author is now aware of any  $\lambda$ -structure on the *higher Burnside ring*. However, Siebeneicher defined a  $\lambda$ -structure on the Burnside ring of a finite group  $A(G)$  in [52] by means of  $\lambda^k : A(G) \rightarrow A(G)$ , which sends a  $G$ -set  $S \mapsto \{T \subset S \mid |T| = k\}$  with its canonical  $G$ -action.

**Proposition 4.9.** *For a finite group  $G$ , the operations  $\omega^k : G_0(G_+) \rightarrow G_0(G_+)$  induced by those in Theorem 4.7 define the same  $\lambda$ -structure on  $G_0(G_+) \cong A(G)$  as that of Siebeneicher.*

*Proof.* The assertion follows from Remark 4.3 and the argument in Section 8 of [26]. Note that for any  $S_+ \in \mathbf{FG}(G_+)$ , the ordering on an element of  $\diamond^k S_+$  does not matter up to a  $G_+$ -module isomorphism.  $\square$

**Remark 4.10.** *In general, the  $\lambda$ -operations on  $G_0(G_+) \cong A(G)$  of Siebeneicher gives it only a pre- $\lambda$ -ring structure. However, it becomes a  $\lambda$ -ring if  $G$  is a cyclic group of odd order.*

If  $R = F$  is a field and  $G$  is a finite group, then the category  $\mathcal{P}(F, G)$  is the same as  $\mathbf{FG}(F[G])$ . Using split monomorphisms (resp. isomorphisms) as the cofibrations (resp. weak equivalences) in the Waldhausen categories  $\mathcal{P}(F, G)$  and  $\mathbf{FG}(F[G])$  we conclude that they have the same Waldhausen K-theory spectra. Now if, in addition, the order of  $G$  is invertible in  $F$ , then  $F[G]$  is semisimple and the category  $\mathbf{FG}(F[G])$  is the same as the category  $\mathcal{P}(F[G])$ , which is the category of finitely generated and projective modules over  $F[G]$ . Consequently, the K-theory spectrum of  $\mathbf{FG}(F[G])$  is the algebraic K-theory of the group algebra  $F[G]$ . In this manner one can construct ‘illegitimate  $\lambda$ -operations’ on the higher algebraic K-theory of a possibly noncommutative group algebra  $F[G]$  (whenever  $G$  is nonabelian).

**Remark 4.11.** *If  $G$  is a finite abelian group then Grayson’s machinery can be used to directly construct  $\lambda$ -operations on  $K_i(F[G])$ . However, these  $\lambda$ -operations will differ from the ones that we just described above since  $\boxtimes$  is different in the two cases: in the ‘illegitimate case’ it is  $\otimes_F$  with diagonal  $G$ -action, whereas in the other it is  $\otimes_{F[G]}$ .*

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